Stability Analysis of Switched Polynomial Systems using Dissipation Inequalities

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Outline

1. Introduction
2. Problem Formulation
3. Proposed Method:
   A. Polynomial Approach
   B. Dissipation Inequalities
   C. Sum of Squares (sos)
4. Main Result
5. Example
6. Conclusion
1. Introduction (1)

Switched Systems:

- $\dot{x}(t) = f_{\sigma(t)}(x, u, t)$

  where:
  - $f_i$ are vector fields
  - $x$ is the state
  - $u$ is the exogenous input
  - $t$ is the time
  - $\sigma(t) : [0, t_f] \rightarrow \mathbb{Q} \in \{0, 1, 2, \ldots, q\}$ is a piecewise constant function of time = the switching signal
Stability of Switched Systems (see e.g. [Lin and Antsaklis, TAC’08])

- Stability under arbitrary switchings
  - Common quadratic Lyapunov functions ⇒ stability

[Recall: $V(x)$ is a Lyapunov function candidate if it is
  a) differentiable,
  b) positive definite, and
  c) radially unbounded]
1. Introduction (3)

Stability of Switched Systems (cont’d)

- Stability under arbitrary switchings (cont’d)
  - Special cases:
    - state matrices that are pairwise commutative
    - state matrices that are symmetric
    - switched normal systems $A_q A_q^T = A_q^T A_q$
  - Necessary and Sufficient condition [Lin and Antsaklis, CDC’04]
  - Converse Lyapunov Theorem [Dayawansa and Martin, TAC’99]
1. Introduction (4)

Stability of Switched Systems (cont’d)

- Stability under restricted switchings
  - time-domain: dwell-time, average dwell-time, etc. [Hespanha and Morse, CDC’99], [Hespanha, TAC’04]
  - state space: abstractions from partitions of the state-space
    - Multiple Lyapunov functions (MLF) [Peleties-DeCarlo, ACC’91], [Morse, 1997], [Branicky, TAC’98], [Ye-Michel-Hu, TAC’98]

  [Recall: The intrinsic discontinuous nature of hybrid systems:
  ⇒ use multiple Lyapunov function to form a single Lyapunov function].
Outline

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2. Problem Formulation
3. Proposed Method:
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4. Main Result
5. Example
6. Conclusion
2. Problem Formulation

Stability of Switched Systems under arbitrary switchings

- Finding a common Lyapunov function is not an easy task.
- Problems are induced by discontinuities of switchings
- There is no systematic way of finding a Lyapunov function (in general)

Problem:
Find a common Lyapunov function for the stability analysis of switched systems.
Outline

1. Introduction
2. Problem Formulation
3. Proposed Method:
   A. Polynomial Approach
   B. Dissipation Inequalities
   C. Sum of Squares (sos)
4. Main Result
5. Example
6. Conclusion
3. Proposed Method

Problem:
Find a common Lyapunov function for the stability analysis of switched systems:

Proposed Method:

- Step 1. Transform the switched system into a polynomial system
- Step 2. Establish stability Lyapunov condition using dissipation inequalities
- Step 3. Obtain the sum-of-squares sos and compute a polynomial Lyapunov function using semidefinite programming
Outline

1. Introduction
2. Problem Formulation
3. Proposed Method:
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4. Main Result
5. Example
6. Conclusion
3.A. Polynomial Approach (1)

**Polynomial Representation:**

- Given a switched system, there exists a unique polynomial $P$ of order $q$ such that $f_i(x, u) = P(x, u, i), \forall i \in \{0, 1, \ldots, q\}$
- The polynomial $P$ is given by
  $$P(x, u, s) = \sum_{k=0}^{q} f_k(x, u) L_k(s) \text{ and } Q(s) = 0$$
  where:
  - $L_k(s) = \prod_{i=0 \atop i \neq k}^{q} \frac{s - i}{k - i}$ are the Lagrange polynomial interpolation quotients
  - $Q(s) = \prod_{k=0}^{q} (s - k)$ is the constraint polynomial

47th IEEE CDC 2008 – p.12
Proposition 1 Given a switched system $\dot{x}(t) = f_{\sigma(t)}(x, u, t)$ with a drift vector field $L(s) = [l_0(s), l_1(s), ..., l_q(s)]^T$ then there exists:

- an unique polynomial system with the polynomial state equation $P(x, u, s)$ of order $q + 1$, and
- a constraint algebraic polynomial $Q(s)$ such that:

$$
\dot{x} = P(x, u, s) \\
0 = Q(s)
$$
3.A. Polynomial Approach (3)

Examples:

- **n=2 sub-systems:**
  \[ \dot{x} = f_0(x)(1-s) + f_1(x)s \]
  \[ s = \{0, 1\} \]
  \[ Q(s) = s(s - 1) = 0 \text{ [constraint polynomial]} \]

- **n=3 sub-systems:**
  \[ \dot{x} = f_0(x)(s - 1)(s - 2) + f_1(x)s(2 - s) + \frac{f_2(x)(s)(s - 1)}{2} \]
  \[ s = \{0, 1, 2\} \]
  \[ Q(s) = s(s - 1)(s - 2) = 0 \text{ [constraint polynomial]} \]
Outline

1. Introduction
2. Problem Formulation
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6. Conclusion
3.B. Dissipation Inequalities (1)

- A new alternative for finding Lyapunov function:
  - Use of **dissipation equations** in order:
    - to avoid difficulties of discontinuities
    - to compute effectively the solution

- **Dissipation inequalities** are considered as **generalized Lyapunov-like inequalities** or **energy-like inequalities**.
Dissipation inequality: See e.g., [Ebenbauer-Allgöwer 2006]

**Definition 1** Given a system of the form \( f(x, \dot{x}, u, y) \), and given a positive semidefinite function \( V \). Then an inequality of the form

\[
\dot{V} \leq a(x, \dot{x}, u, y)
\]

is called **dissipation inequality**.

- \( V \) is called the **storage function**
- \( a \) (scalar-valued function) is called the **supply rate**
3.B. Dissipation Inequalities (3)

Stability Theorem:

**Theorem 1** The equilibrium point $x = 0$ of $f(x, \dot{x}, u, t) = 0$ is *stable* if there exist $V, \lambda, Q(s)$ such that

$$\nabla V(x) \dot{x} \leq \|Q(s)\|^2 \lambda(x, s)$$

is satisfied for some neighborhood of the origin.

For the equivalent polynomial representation, the inequality becomes

$$\nabla V(x) \left( \sum_{k=0}^{q} f_k(x, u) l_k(s) \right) \leq \left\| \prod_{k=0}^{q} (s - k) \right\|^2 \lambda(x, s)$$
3.B. Dissipation Inequalities (4)

- In general, it is difficult to find a Lyapunov function $V$

- However, methods based on semidefinite programming and sum-of-squares (sos) decomposition will allow the designer to verify Lyapunov inequalities

- We assume that functions $V, \lambda, Q(s)$ are polynomials
Outline

1. Introduction
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5. Example
6. Conclusion
3.C. Sum of Squares sos (1)

Sum of Squares, sos:

**Definition 2** [Parrilo 2000], [Papachristodoulou-Prajna, 2005] For \( x \in \mathbb{R}^n \), a multivariate polynomial \( p(x) \) is a sum of squares (sos) if there exist some polynomials \( r_i(x) \), \( i = 1, ..., M \) such that

\[
p(x) = \sum_{i=1}^{M} r_i^2(x)
\]

• ⇒ If \( p(x) \) is a sos then \( p(x) \geq 0 \), for all \( x \in \mathbb{R}^n \)
3.C. Sum of Squares sos (2)

Sum of Squares sos: (cont’d)

**Proposition 2** [Parrilo 2000]
A polynomial $p(x)$ of degree $2d$ is a sos if and only if there exists a positive semidefinite matrix $Q$ and a vector of monomials $Z(x)$ containing monomials in $x$ of degree $\leq d$ such that

$$p(x) = Z(x)^T Q Z(x)$$

**Proposition 3** [Parrilo 2000]
Given a polynomial $V(x)$ of degree $2d$, let

$$\varphi(x) = \sum_{i=1}^{n} \sum_{j=1}^{d} \epsilon_{i,j} x_i^{2j}$$

such that $\sum_{j=1}^{d} \epsilon_{i,j} > \gamma \quad \forall i = 1, ..., n,$ with $\gamma$ a positive number, and $\epsilon_{i,j} \geq 0$ for all $i$ and $j$. Then the condition

$$V(x) - \varphi(x)$$

is a sos guarantees the positive definiteness of $V(x)$. 
Outline

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4. Main Result
5. Example
6. Conclusion
4. Main Result: Problem Reformulation

**Proposition 4** The dissipation inequality problem

\[ \nabla V(x) \left( \sum_{k=0}^{q} f_k(x,u) l_k(s) \right) \leq \| Q(s) \|_2^2 \lambda(x,s) \]

can now be reformulated as follows:

\[ \Rightarrow \text{New Problem: Find a polynomial } V(x) \text{ such that} \]

\[ V(x) - \varphi(x) \text{ is sos} \]
\[ -\nabla V(x) (f_0(x)(1-s) + f_1(x)s) + \|s(1-s)\|_2^2 \lambda(x,s) \text{ is sos} \]

**Solution:**

- \[ \Rightarrow \text{the polynomials } V(x), \lambda(x,s), \text{and the positive definite function} \]
  \[ \varphi(x) \text{ can be computed using SOSTOOLS [Papachristodoulou et al. 2002]} \]
Outline

1. Introduction
2. Problem Formulation
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4. Main Result
5. Example
6. Conclusion
5. Example (1)

- Consider the set of systems described by the drift vector field
- \( F(x) = [f_0(x)f_1(x)] \) where

\[
\begin{align*}
  f_0(x) &= \begin{bmatrix}
    x_1^3 + 2x_1x_2^2 + 2x_1^2x_2 + 4x_2^3 \\
    -x_1^3 - 2x_1x_2^2 - 3x_1^2x_2 - 6x_2^3
  \end{bmatrix} \\
  f_1(x) &= \begin{bmatrix}
    -2x_1^3 - 4x_1x_2^2 + x_1^2x_2 + 2x_2^2 \\
    -0.5x_1^3 - x_1x_2^2 + 3x_1^2x_2 + 6x_2^3
  \end{bmatrix}
\end{align*}
\]

- The polynomial equivalent representation:
  \[ \dot{x} = f_0(x)(1 - s) + f_1(x)s \]
  \[ s = \{0, 1\} \]
  \[ Q(s) = s(s - 1) = 0 \text{ [constraint polynomial]} \]
5. Example (2)

- Polynomial Representation
- Dissipation Inequalities
  - \[ V(x) - \varphi(x) \geq 0 \text{ is sos} \]
  - \[ -\frac{\partial V}{\partial x} (f_0(1 - s) + f_1s) + (s^4 - 2s^3 + s^2)\lambda(x, s) \text{ is sos} \]

- Using MATLAB Toolbox SOSTOOLS:
  - \[ a \text{ Lyapunov function of degree 6 is found} \]
  - so that the equilibrium point is GUAS \[ \Rightarrow \text{ it is stable.} \]

- NB. Two other examples are given in the paper (cf. the proceedings).
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4. Main Result
5. Example
6. Conclusion
6. Conclusion (1)

Summary:

- An alternative approach to study the stability of switched systems under arbitrary switchings.
- It is based on the following method:
  1. Transform the switched system into a polynomial system.
  2. Establish the stability Lyapunov condition using dissipation inequalities.
  3. Solve the sum-of-squares, sos decomposition using MATLAB Toolbox SOSTOOLS.
6. Conclusion (2)

References:

6. Conclusion (3)

Future work:
- Exploit other methods based on dissipation inequalities for stability analysis of switched systems
- Extension of the approach to the stability analysis of non-linear non-polynomial switched systems
- Study the stability analysis of switched systems under restricted switchings
Merci pour votre attention !
Future Work: Master Thesis Proposals

Joint Master Thesis Proposals: 9 months at UniAndes + 3 months at EMN France:

- **Master Thesis Subject N.1:**
  - Stability analysis of discrete-time switched systems
  - Advisors: Nicanor Quijano and N. Rakoto

- **Master Thesis Subject N.2:**
  - Evolutionary game theory for switched discrete-event systems control
  - Advisors: Nicanor Quijano and N. Rakoto
Future Work: Master Thesis Proposals

Joint Master Thesis Proposals: 9 months at UniAndes + 3 months at EMN France:
(cont’d)

- Master Thesis Subject N.3:
  - Model Predictive Control (MPC) of supply chain management systems
  - Advisors: Fidel Torres and N. Rakoto