Switched Systems: state estimation and switching function estimation

Michel FLIESS, Cédric JOIN and Wilfrid PERRUQUETTI

Projet ALIEN, INRIA
E-mail: Michel.Fliess@polytechnique.edu, Cedric.Join@cran.uhp-nancy.fr, wilfrid.perruquetti@ec-lille.fr.

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Switched systems may be viewed as higher–level abstractions of hybrid systems, obtained by neglecting the details of the discrete behavior.

\[
\dot{x} = f_i(x, t, u), \quad x \in \mathcal{X} \subset \mathbb{R}^n, \quad u \in \mathbb{R}^m, \quad i \in \Gamma_M \triangleq \{1, \ldots, M\}.
\]

- at each instant one and only one subsystem is active,
- selection may depend on time, state or external operator choice,
- rule to switch.
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\dot{x} = f_i(x, t, u), \; x \in \mathcal{X} \subset \mathbb{R}^n, \; u \in \mathbb{R}^m, \quad (1)
\]

\[
f_i : \mathcal{X} \subset \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^m \rightarrow \mathbb{R}^n, \quad i \in \Gamma_M \triangleq \{1, \ldots, M\}.
\]

- at each instant one and only one subsystem is active,
- selection may depends on time, state or external operator choice,
- rule to switch.
Informally, a switched system is composed of a family of dynamical subsystems (linear or nonlinear) \( \{DS_i\} \), and a rule, called the switching law, that orchestrates the switching between them.

\[
\dot{x} = f_{\sigma(x,t)}(x, t, u), \quad x \in \mathcal{X} \subset \mathbb{R}^n, \quad u \in \mathbb{R}^m, \quad \{DS_i\},
\]

\( \sigma(x, t) : \mathbb{R}^n \times \mathbb{R} \rightarrow \Gamma_M \) is the switching law.

A switched system is linear when all sub systems are linear this is when \( f_{\sigma(x,t)}(x, t, u) \) is linear. The system reads as

\[
\dot{x} = A_{\sigma(x,t)}x + B_{\sigma(x,t)}u.
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Increasing interest in the control problems of switched systems:

- **stability** [Agrachev et al. 01, Boscain 02, Branicky 94, Branicky 98, Hespanha et al. 99, Liberzon et al. 99, Vu et al. 9, Mancilla-Aguilar et al. 00, Pettersson et al. 96, Skafidas et al. 99] (with many applications see, for example, [Buisson et al. 05] for application to electrical power converters),

- **stabilization** [Bourdais et al. 06, Moulay et al. 07, Persisy et al. 03, Persisy et al. 04, Pettersson 99, Wicks et al. 94, Wicks et al. 97, Wicks et al. 98, Zhai et al. 03, Wang et al. 04],
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- tracking [Bourdais et al. 07],
- controllability results [Sun et al. 02, Xie et al. 02],
input-to-state properties, . . . .

See, e.g., [Branicky 93, Brockett et al. 93, Liberzon et al. 99, Liberzon et al. 03, Sun et al. 05] for a survey of this type of results.

- Existing results need a complete description of all $DS_i$
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Problem formulation : an overview

Fact : need to know $\sigma$

Most of the existing results need the **knowledge of the switching function** $\sigma$.

**Observability and state estimation** is a key problem for such systems, where discrete and continuous parts are mixed.

In order to reconstruct the state one has to know **which dynamics** is active.
Consider a finite set of ordinary differential equations (ODE)

\[ \dot{x} = f_i(t, x, u), \quad (3) \]

where \( f_i : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n, \quad i \in \Gamma_M \triangleq \{1, \ldots, M\}. \)

knowledge of \( y = h(x), \) on \([t_1, t] \) \( \Rightarrow \) estimation of \( x(t), \sigma(t) \) or equivalently the current active sub-model
Existing results concerning this problem:

- Generic setting for the observability of switched linear systems in continuous setting has been given in [Babaali et al. 03].

- Observability of switched linear systems in the case of deterministic switching, [Alessandri et al. 01, Vidal et al. 02, Vidal et al. 03]

- Unobserved switching case was analyzed in [De Santis et al. 03].
Problem formulation : an overview

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Problem formulation: an overview

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Problem formulation: examples and remarks

All we want to know is in the output signal...

Let us consider

\[
\begin{align*}
\dot{x}_1 &= x_2 + \sigma(t)x_1 \\
\dot{x}_2 &= u(t) - 2\sigma(t)x_1 - x_2 \\
y &= x_1
\end{align*}
\]

\[
\ddot{y} = u - 2\sigma y - \dot{y}
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Problem formulation : examples and remarks

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Problem formulation: examples and remarks

All we want to know is in the output signal...

$$\sigma = \frac{\ddot{y} + \dot{y} - u}{\dot{y} - y}$$

$$x_1 = y$$

$$x_2 = \dot{y} - \sigma y$$
Problem formulation: examples and remarks

All we want to know is in the output signal...

Our point of view

Obtained model relies on real-time estimations of derivatives for noisy signals

\[ y(t) = a_0 + a_1 t + a_2 t^2 + \ldots + a_N t^N \]
\[ y^{(N+1)} = 0 \]
Problem formulation : examples and remarks

All we want to know is in the output signal...

Our point of view

Obtained model relies on real-time estimations of derivatives for noisy signals.
It is clear that if one knows which current ODE is active (the current index "i") then one can say something about the current continuous state.

Thus a key problem is the real time computation of the so-called switching signal defined by

$$\sigma(t, x) : \mathbb{R} \times \mathbb{R}^n \rightarrow I_M$$

$$(t, x) \mapsto \sigma(t, x)$$

where $\sigma(t, x) \in I_M = \{1, \ldots, M\}$ corresponds to the index associated with the current active ODE.
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Problem formulation : examples and remarks

Distinguishability : an intuitive flavor

Remark

Let us consider a pair of such LODE
\[ \dot{x} = A_1 x + B_1 u, \dot{x} = A_2 x + B_2 u. \]
One can distinguish the two LODE if for any non trivial input the two systems produce two different outputs (roughly speaking).

On this basis, one can encounter the following facts:

- a pair of LODE may be strongly indistinguishable: any non trivial input will produce for the two systems the same output,
- a pair of LODE may be weakly indistinguishable: some particular inputs will produce the same outputs,
Problem formulation: examples and remarks

Example of indistinguishability

A pair of LODE may be strongly indistinguishable: any non trivial input will produce for the two systems the same output. The two following LODE

\[
\begin{align*}
\ddot{y}_1 + 3\dot{y}_1 + 2y_1 &= \dot{u}_1 + 2u_1, \\
\ddot{y}_2 + 4\dot{y}_2 + 3y_2 &= \dot{u}_2 + 3u_2,
\end{align*}
\]

always produce the same output (if starting from the same initial condition) whatever is the applied input (see section 1 for details). It can be seen on this example that the two system has simply different non observable mode.
Problem formulation : examples and remarks

Example of indistinguishability

![Graph showing example of indistinguishability](image)
A pair of LODE may be weakly indistinguishable: some particular inputs will produce the same outputs: the two systems are indistinguishable.

The two following LODE

\[
\begin{align*}
\dot{y}_1 &= u_1, \\
\ddot{y}_2 + \dot{y}_2 &= 2u_2,
\end{align*}
\]

it is clear that if the input of these systems is \(u_1 = u_2 = \exp(t)\) then the two systems have a common trajectory namely \(y(t) = \exp(t)\): a particular input may give rise to the same output behaviour for two different LDOE, the two systems can not be distinguish for this singular input.
Problem formulation : examples and remarks

Example of indistinguishability
It is thus important to characterize such singular inputs and then to give efficient tools for observations.
Some examples and remarks

Except for these pathological cases, the state (continuous and discrete or the switching signal) reconstruction of an hybrid system is a three stage process as follows:

1. using the measured output $y$ (and eventually the input $u$) one needs to reconstruct $y, \dot{y}, \ldots, y^{(k_y)}$ up to some finite order $k_y$ which is not necessarily equal to the state dimension ($k_y$ may be smaller) and eventually $u, \dot{u}, \ldots, u^{(k_m)}$,

2. reconstruct the switching signal or the discrete state: a set of signal are constructed using the obtained reconstructed signals $y, \dot{y}, \ldots, y^{(k_y)}; u, \dot{u}, \ldots, u^{(k_m)}$ in order to distinguish the active subsystem.

3. reconstruct the continuous state: using for example step 1 or eventually a refined technic using the knowledge of the actual active subsystem.
Consider switched linear systems of the form:

\[
\dot{x} = A_{\sigma(t)}x + B_{\sigma(t)}u, \\
y = C_{\sigma(t)}x + D_{\sigma(t)}u,
\]

\[x \in \mathbb{R}^n, u \in \mathbb{R}^m, y \in \mathbb{R}^p, \quad (7)\]

where \(\sigma(t)\) is the switching signal taking value within the index set \(I_M = \{1, ..., M\}\).

In the rest we address the problem of the reconstruction of the switching signal \(\sigma(t)\) in “real-time”, and of the state variables if it is possible.
Problem formulation : an Input-output behavior

Consider switched linear systems of the form:

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where \( \sigma(t) \) is the switching signal taking value within the index set \( I_M = \{1, \ldots, M\} \).

In the rest we address the problem of the reconstruction of the switching signal \( \sigma(t) \) in “real-time”, and of the state variables if it is possible.
From the problem formulation and the above mentioned examples, it is clear that for each \( i \in I \) we only need an input/output representation of our linear subsystems. In the rest we adopt a matrix transfer representation of the input/output behavior:

\[
\alpha_i \left( \frac{d}{dt} \right) y_i = \beta_i \left( \frac{d}{dt} \right) u, \quad i \in I. \tag{8}
\]
Distinguishability
The monovariable case

Consider two monovariable linear systems with transfer functions

\[ \frac{b_i}{a_i}, \ i = 1, 2, \]

with \( a_i, b_i \in \mathbb{R}[s], b_i \neq 0, (a_i, b_i) = 1. \) It is clear that they do not exhibit the same input-output behavior if the two transfer functions are different, i.e.,

\[ a_1 b_2 - a_2 b_1 \neq 0. \]

There might exist nevertheless particular inputs for which the two outputs coincide.
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There might exist nevertheless particular inputs for which the two outputs coincide.
Assume that two input-output systems satisfy $a_i \left( \frac{d}{dt} \right) y_i = b_i \left( \frac{d}{dt} \right) u$, $i = 1, 2$. It is clear that we cannot distinguish their input-output behaviors if, and only if, $u$ and $y = y_1 = y_2$ satisfy the matrix differential equation

$$
\begin{pmatrix}
  a_1 & -b_1 \\
  a_2 & -b_2
\end{pmatrix}
\begin{pmatrix}
  y \\
  u
\end{pmatrix} = 
\begin{pmatrix}
  0 \\
  0
\end{pmatrix}.
$$

(9)

Classic algebraic manipulations show that Eq. (9) is equivalent to

$$
\mathcal{A} \left( \frac{d}{dt} \right) u = 0 \quad \mathcal{B} \left( \frac{d}{dt} \right) y = 0.
$$

(10)

where $\mathcal{A}, \mathcal{B} \in \mathbb{R}[\frac{d}{dt}]$, $\mathcal{A}\mathcal{B} \neq 0$. (see next result)
Distinguishability
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$$\begin{pmatrix} a_1 & -b_1 \\ a_2 & -b_2 \end{pmatrix} \begin{pmatrix} y \\ u \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$  \hfill (9)

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$$\mathcal{A} \left( \frac{d}{dt} \right) u = 0 \quad \mathcal{B} \left( \frac{d}{dt} \right) y = 0.$$  \hfill (10)

where $\mathcal{A}, \mathcal{B} \in \mathbb{R}\left[ \frac{d}{dt} \right]$, $\mathcal{A} \mathcal{B} \neq 0$. (see next result)
Distinguishability
The monovariable case: definition

Definition

The two systems are said to be strongly distinguishable if, and only if, \( A \) and \( B \) are constant. This is equivalent saying that the two systems have the same input-output behavior only for \( u = y_1 = y_2 = 0 \). If not the two systems are said to be weakly distinguishable.

The next result summarizes the above computations.
Definition

The two systems are said to be \textit{strongly distinguishable} if, and only if, $\mathbf{A}$ and $\mathbf{B}$ are constant. This is equivalent saying that the two systems have the same input-output behavior only for $u = y_1 = y_2 = 0$. If not the two systems are said to be \textit{weakly distinguishable}.

The next result summarizes the above computations.
Distinguishability
The monovariable case: result

**Theorem**

In (10) $\mathcal{A}$ and $\mathcal{B}$ are given by

$$
\mathcal{A} = \gcd(b_1 p_1^1, b_2 p_2^2),
\mathcal{B} = (a_2 b_1' - a_1 b_2'),
$$

(11)

where $b = \gcd(b_1, b_2)$, $b_1 = b b_1'$, $b_2 = b b_2'$, $a = \gcd(a_1, a_2)$, $a_1 = a a_1'$, $a_2 = a a_2'$, $p_1 = \text{lcm}(a_1', a_2' b_1' - a_1' b_2')$, and $p_2 = \text{lcm}(a_2', a_2' b_1' - a_1' b_2')$.

**Strong distinguishability** is equivalent to $(a_2 b_1' - a_1 b_2') \in \mathbb{R} \setminus \{0\}$ and $\gcd(b_1 p_1^1, b_2 p_2^2) \in \mathbb{R} \setminus \{0\}$. 
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Distinguishability
The monovariable case: proof

Proof

Using \( b = \gcd(b_1, b_2) \) and \( a = \gcd(a_1, a_2) \), (9) reads as:

\[
\begin{pmatrix}
  a a' & -b b' \\
  a a' & -b b'
\end{pmatrix}
\begin{pmatrix}
  y \\
  u
\end{pmatrix}
=
\begin{pmatrix}
  0 \\
  0
\end{pmatrix}.
\] (12)

Let us recall that \((a_i, b_i) = 1\) implies \((a, b) = (a'_1, b'_1) = (a'_2, b'_2) = 1\). Replacing the 2nd line (L2) by \(b'_1L2 - b'_2L1\) leads to

\[
\begin{pmatrix}
  a a' & -b b' \\
  a (a'_2 b'_1 - a'_1 b'_2) & 0
\end{pmatrix}
\begin{pmatrix}
  y \\
  u
\end{pmatrix}
=
\begin{pmatrix}
  0 \\
  0
\end{pmatrix}.
\]
Distinguishability
The monovariable case: proof (cont.)

Proof

Replacing the 1st line \((L1)\) by \(p_1^1 L1 - p_2^1 L2\) leads to

\[
\begin{pmatrix}
0 & -b b' p_1^1 \\
\alpha (a'_2 b'_1 - a'_1 b'_2) & 0
\end{pmatrix}
\begin{pmatrix}
y \\
u
\end{pmatrix}
= \begin{pmatrix}
0 \\
0
\end{pmatrix}
\]

\[b_1 \left( \frac{d}{dt} \right) p_1^1 \left( \frac{d}{dt} \right) u = 0 \quad (13)\]

\[
\left[ a_1 \left( \frac{d}{dt} \right) b'_2 \left( \frac{d}{dt} \right) - a_2 \left( \frac{d}{dt} \right) b'_1 \left( \frac{d}{dt} \right) \right] y = 0 \quad (14)
\]
Proof

Starting with (12) and replacing the 1st line (L1) by $b'_1 L2 - b'_2 L1$ leads to

$$\begin{pmatrix} a (a'_2 b'_1 - a'_1 b'_2) & 0 \\ a a'_2 & -b b'_2 \end{pmatrix} \begin{pmatrix} y \\ u \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Replacing the 2nd line (L2) by $p^2_1 L2 - p^2_2 L1$ leads to

$$\begin{pmatrix} a (a'_2 b'_1 - a'_1 b'_2) & 0 \\ 0 & -b b'_2 p^2_1 \end{pmatrix} \begin{pmatrix} y \\ u \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
Distinguishability
The monovariable case: proof (end)

Proof

\[ b_2 \left( \frac{d}{dt} \right) p_1^2 \left( \frac{d}{dt} \right) u = 0 \]  \hspace{1cm} (15)

\[ \left[ a_1 \left( \frac{d}{dt} \right) b'_2 \left( \frac{d}{dt} \right) - a_2 \left( \frac{d}{dt} \right) b'_1 \left( \frac{d}{dt} \right) \right] y = 0 \]  \hspace{1cm} (16)

The conclusion follows.
Distinguishability

The monovariable case

Remark

The two systems are said to be strongly distinguishable if, and only if, the only solution to (13)–(15)–(14) is the trivial one that is $y = u = 0$. This is equivalent saying that the two systems have the same input-output behavior only for $u = y_1 = y_2 = 0$. If not the two systems are said to be weakly distinguishable.
Distinguishability
Scalar first order systems

Let us consider (7) with \( n = 1, m = p = 1 \) and \( M = 2 \): for a given \( i \in I \) the corresponding system has a transfer function of the form

\[
F_i(s) = \frac{k_i}{1 + \tau_i s}, \quad k_i = \frac{b_i}{a_i}, \quad \tau_i = \frac{1}{a_i},
\]

(17)

the two transfer functions are different iff

\[
k_1(1 + \tau_2 s) - k_2(1 + \tau_1 s) \neq 0.
\]

(18)

Nevertheless for the two systems some input/output behavior cannot be distinguish iff

\[
\begin{pmatrix}
(1 + \tau_1 s) & -k_1 \\
(1 + \tau_2 s) & -k_2 \\
\end{pmatrix}
\begin{pmatrix}
y \\
u \\
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
\end{pmatrix}.
\]

(19)
Distinguishability
Scalar first order systems: Case a) $k_1 \tau_2 = k_2 \tau_1$ and $k_1 \neq k_2$

Thus (19) has a unique singular behavior for which the two systems cannot be distinguish

$$u = 0, y = 0,$$  \hspace{1cm} (20)

which means that, when at rest, the two systems cannot be distinguished: they are strongly distinguishable.
Distinguishability
Scalar first order systems: Case a) \( k_1 \tau_2 = k_2 \tau_1 \) and \( k_1 \neq k_2 \)

This result comes directly from Theorem 2: since \( a_1 = 1 + \tau_1 s, b_1 = k_1, a_2 = 1 + \tau_2 s, b_2 = k_2 \) we have \((a_2' b_1' - a_1' b_2') = k_1 - k_2\), it comes out that \( p^1 = \text{lcm}(a_1', a_2' b_1' - a_1' b_2') = (1 + \tau_1 s)\), \( p^2 = \text{lcm}(a_2', a_2' b_1' - a_1' b_2') = (1 + \tau_2 s)\) and thus Eq (10) and (11) reads as

\[
\gcd(b_1 p_1^1, b_2 p_2^2) u = 0 \iff u = 0,
\]

\[
(a_2 b_1' - a_1 b_2') y = 0 \iff (k_1 - k_2) y = 0,
\]

leading to the following singular i/o: \( y = 0, u = 0 \).
The monovariable case

Examples

The multivariable case

Distinguishability

Scalar first order systems: Case b) \( k_1 \tau_2 - k_2 \tau_1 \neq 0 \)

Thus (19) has a unique singular behavior for which the two systems can not be distinguish

\[
0 = (k_1 \tau_2 - k_2 \tau_1) \dot{y} + (k_1 - k_2)y, \tag{22}
\]

\[
u = \frac{\tau_1 \dot{y} + y}{k_1} = \frac{\tau_2 \dot{y} + y}{k_2}, \tag{23}
\]

solutions of (22) are of the form \( y = c \exp(\alpha t), \alpha = \frac{-(k_1 - k_2)}{(k_1 \tau_2 - k_2 \tau_1)} \)

which include the case \( c \) (constant) when \( k_1 = k_2 \); moreover (23) gives \( u = c \frac{\tau_1 \alpha + 1}{k_1} \exp(\alpha t) \) leading to \( u = d \exp(\alpha t) \) with

\[
d = c \frac{(\tau_2 - \tau_1)}{(k_1 \tau_2 - k_2 \tau_1)}. \]

Thus (19) has a unique singular behavior for which the two systems can not be distinguish \( y = c, u = \frac{c}{k_1} \) when \( k_1 = k_2 \) or \( y = c \exp(\alpha t), u = d \exp(\alpha t), \) when \( k_1 \neq k_2 \).
Distinguishability
Scalar first order systems : Case b) \( k_1 \tau_2 - k_2 \tau_1 \neq 0 \)

This can be obtained directly using Theorem 2.

- When \( k_1 = 2, \tau_1 = 1, k_2 = 1, \tau_2 = 2 \): \( k_1 \tau_2 - k_2 \tau_1 = 3 \neq 0 \), one has

  \[ a_1 = s + 1, b_1 = 2, a_2 = 2s + 1, b_2 = 1 \]

from which one gets

  \[ b'_1 = 2, b'_2 = 1, a_1 = a'_1, a_2 = a'_2, \]

and finally:

  \[ (a'_2 b'_1 - a'_1 b'_2)) = 3s + 1. \tag{24} \]

Thus \( p^1 = \text{lcm}(s + 1, 3s + 1) = (s + 1) (3s + 1) \) and

\( p^2 = \text{lcm}(2s + 1, 3s + 1) = (2s + 1) (3s + 1) \) leads to

\[ p^1 = p^1 a'_1 = p^1_2 (a'_2 b'_1 - a'_1 b'_2), p^1 = (3s + 1), p^1_2 = (s + 1) \]

\[ p^2 = p^2 a'_2 = p^2_2 (a'_2 b'_1 - a'_1 b'_2), p^2 = (3s + 1), p^2_2 = (2s + 1) \]

Eq (10) and (11) reads as

\[ \gcd(b_1 p^1, b_2 p^2) u = 0 \iff (3s + 1) u = 0, \]

\[ (a_2 b'_1 - a_1 b'_2) y = 0 \iff (3s + 1) y = 0, \tag{25} \]

leading to the following singular i/o:

\[ y = y_0 \exp(-t/3), \]
Distinguishability
Scalar first order systems: Case b) \( k_1 \tau_2 - k_2 \tau_1 \neq 0 \)

- When \( k_1 = 1, \tau_1 = 1, k_2 = 2, \tau_2 = 1 : k_1 \tau_2 - k_2 \tau_1 = -1 \neq 0 \), one has
  \( a_1 = s + 1, b_1 = 1, a_2 = s + 1, b_2 = 2 \)
  from which one gets
  \( b'_1 = 1, b'_2 = 2, a_1 = a'_1 = a_2 = a'_2 \), and finally
  \[
  (a'_2 b'_1 - a'_1 b'_2) = -(s + 1) .
  \] (26)

Thus \( p^1 = \text{lcm}(s + 1, s + 1) = (s + 1) \) and
\( p^2 = \text{lcm}(s + 1, s + 1) = (s + 1) \) leads to

\[
\begin{align*}
p^1 &= p^1 a'_1 = p^1_2 (a'_2 b'_1 - a'_1 b'_2), p^1_1 = 1, p^1_2 = -1, \\
p^2 &= p^2 a'_2 = p^2_2 (a'_2 b'_1 - a'_1 b'_2), p^2_1 = 1, p^2_2 = -1,
\end{align*}
\]

Eq (10) and (11) reads as

\[
\begin{align*}
gcd(b_1 p_1^1, b_2 p_1^2) u &= 0 \iff u = 0, \\
(a_2 b'_1 - a_1 b'_2) y &= 0 \iff (s + 1) y = 0,
\end{align*}
\] (27)

leading to the following singular i/o: \( u = 0, y = y_0 \exp(-t) \).
Let us consider (7) with $n = 2, m = p = 1$ and $M = 2$: for a given $i \in I$ the corresponding system has a transfer function of the form

$$F_i(s) = \frac{b_{1,i}s + b_{0,i}}{s^2 + a_{1,i}s + a_{0,i}}, \quad (28)$$

the two transfer functions are different iff

$$(s^2 + a_{1,1}s + a_{0,1})(b_{1,2}s + b_{0,2}) - (s^2 + a_{1,2}s + a_{0,2})(b_{1,1}s + b_{0,1}) \neq 0. \quad (29)$$
Nevertheless for the two systems some input/output behavior cannot be distinguish iff

\[
\begin{pmatrix}
(s^2 + a_{1,1}s + a_{0,1}) & -(b_{1,1}s + b_{0,1}) \\
(s^2 + a_{1,2}s + a_{0,2}) & -(b_{1,2}s + b_{0,2})
\end{pmatrix}
\begin{pmatrix}
y \\
u
\end{pmatrix}
= \begin{pmatrix}
0 \\
0
\end{pmatrix}
\] (30)
Let us investigate some numerical examples:

- When \( a_1 = s^2 + s + 1, b_1 = (s + 1), a_2 = s^2 + s + 1, b_2 = (s + 2) \), one obtains \( b'_1 = (s + 1), b'_2 = (s + 2), a'_1 = a'_2 = 1 \) and

\[
(a'_2 b'_1 - a'_1 b'_2) = -1.
\]

Thus \( p^1 = \text{lcm}(1, -1) = -1 \) and \( p^2 = \text{lcm}(1, -1) = -1 \) leads to

\[
p^1 = p^1 a'_1 = p^1_2 (a'_2 b'_1 - a'_1 b'_2), p^1_1 = -1, p^1_2 = 1
\]

\[
p^2 = p^2 a'_2 = p^2_2 (a'_2 b'_1 - a'_1 b'_2), p^2_1 = -1, p^2_2 = 1
\]

Eq (10) and (11) reads as

\[
gcd(b_1 p^1_1, b_2 p^2_1) u = 0 \iff gcd((s + 1), (s + 2)) u = 0
\]

\[
(a_2 b'_1 - a_1 b'_2) y = 0 \iff (s^2 + s + 1) y = 0
\]

leading to the following singular i/o:

\[
y(t) = \exp \left( -\frac{t}{2} \right) \left( y_0 \cos \left( \frac{\sqrt{3}}{2} t \right) + \frac{(y_0 + \frac{1}{2} y_0)}{\sqrt{3}} \sin \left( \frac{\sqrt{3}}{2} t \right) \right).
\]
Distinguishability
Scalar second order systems

• When \( a_1 = s^2 + s + 1, b_1 = (s + 1), a_2 = s^2 + s + 2, b_2 = (s + 1) \) one obtains \( b'_1 = 1, b'_2 = 1, a'_1 = s^2 + s + 1, a'_2 = s^2 + s + 2 \) and

\[
(a'_2 b'_1 - a'_1 b'_2) = 1. \tag{33}
\]

Thus \( p^1 = \text{lcm}(s^2 + s + 1, 1) = s^2 + s + 1 \) and \( p^2 = \text{lcm}(s^2 + s + 2, 1) = s^2 + s + 2 \) leads to

\[
p^1 = p^1 a'_1 = p^2(a'_2 b'_1 - a'_1 b'_2), p^1 = 1, p^2 = s^2 + s + 2
\]

\[
p^2 = p^2 a'_2 = p^2(a'_2 b'_1 - a'_1 b'_2), p^2 = 1, p^2 = s^2 + s + 2
\]

Eq (10) and (11) reads as

\[
\gcd(b_1 p^1_1, b_2 p^2_2)u = 0 \Leftrightarrow (s + 1)u = 0, \quad (a_2 b'_1 - a_1 b'_2) y = 0, \tag{34}
\]

leading to the following singular i/o : \( u = u_0 \exp(-t), y = 0. \)
Distinguishability
Scalar second order systems

- When \( a_1 = s^2 + s + 1, b_1 = (s + 1), a_2 = s^2 + 2s + 1, b_2 = (s + 1) \) one obtains \( b'_1 = 1, b'_2 = 1, a'_1 = s^2 + s + 1, a'_2 = s^2 + 2s + 1 \), and
  \[
  (a'_2 b'_1 - a'_1 b'_2) = s
  \]  

Thus \( p^1 = \text{lcm}(s^2 + s + 1, s) = s(s^2 + s + 1) \) and
\( p^2 = \text{lcm}(s^2 + 2s + 1, s) = s(s^2 + 2s + 1) \) leads to

\[
\begin{align*}
p^1 &= p^1 a'_1 = p^2(a'_2 b'_1 - a'_1 b'_2), \quad p^1_1 = s, p^1_2 = s^2 + s + 1 \\
p^2 &= p^2 a'_2 = p^2(a'_2 b'_1 - a'_1 b'_2), \quad p^2_1 = s, p^2_2 = s^2 + 2s + 1
\end{align*}
\]

Eq (10) and (11) reads as

\[
\begin{align*}
\gcd(b_1 p^1_1, b_2 p^2_1) u &= 0 \iff s(s + 1) u = 0, \\
(a_2 b'_1 - a_1 b'_2) y &= 0 \iff s y = 0,
\end{align*}
\]

leading to the following singular i/o: \( u = u_0 + u_1 \exp(-t), y = u_0 \) with \( u_0, u_1 \) any constant.
Distinguishability
The multivariable case

Consider two multivariable i/o systems \((\Sigma_i), \; i = 1, 2\), and their transfer matrices \(T_i \in \mathbb{R}(s)^{p \times m}\):

\[
\mathcal{A}_i \begin{pmatrix} y_1 \\ \vdots \\ y_p \end{pmatrix} = \mathcal{B}_i \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix},
\]

\(i = 1, 2\), where

- \(\mathcal{A}_i \in \mathbb{R}(s)^{p \times p}, \; \text{det} \; \mathcal{A}_i \neq 0\),
- \(\mathcal{B}_i \in \mathbb{R}(s)^{p \times m}\),
- \(T_i = \mathcal{A}_i^{-1} \mathcal{B}_i\),
- \(\mathcal{A}_i \) and \(\mathcal{B}_i\) are left coprime.

It is clear that they do not exhibit the same i/o behavior if \(T_1 \neq T_2\) (This will be assumed in the sequel).
Distinguishability
The multivariable case: definition

**Definition**

$(\Sigma_1)$ and $(\Sigma_2)$ are said to be

- **strongly distinguishable** if, and only if, all the different entries in $T_1$ and $T_2$ are strongly distinguishable in the sense of Sect. 1,
- **weakly distinguishable** if not.

🔗 Checking distinguishability results therefore to checking it for several monovariable systems.
Assume from now on that all the subsystems models are known and that any pair is strongly distinguishable.

Let us consider a switching system defined by a finite collection of input/output behaviors driven by LODE satisfying the above given assumptions. As soon as the system is not at rest, for the given control, the measured output can be used to determine which subsystem is active.

From now we want to obtain effective real-time algorithm to determine the current “i”.
Estimation : Switching signal estimation

Overview

If one is able to construct in real time the following quantities

\[ r_i(t) = a_i \left( \frac{d}{dt} \right) y_i - b_i \left( \frac{d}{dt} \right) u, \]

it is clear that the current "i" is such that \( r_i(t) = 0 \) on a sub-set of \( \mathbb{R} \) with non zero measure.

The problem is thus reduced to the real-time computation of time derivative of the output and input despite the noise.
Estimation
Numerical differentiation

The numerical differentiation technics introduced below are of non asymptotic nature, and the desired estimation can be obtained instantaneously (there is a singularity at time $t = 0$). But in practice they are numerically implemented with discrete measured data, thus from a practical point of view, it will be necessary that the sampling time should be small enough with respect to the duration time between two successive switchings$^1$.

\footnote{In practice at least 100 times smaller.}
This algebraic setting for numerical differentiation started in [Fliess et al. 04, Fliess et al. 04]. See [Mboup et al. 07] for further developments, and [Nöthen 07] for interesting discussions and comparisons. Consider a signal $y(t) = \sum_{i=0}^{\infty} y^{(i)}(0) \frac{t^i}{i!}$ which is assumed to be analytic around $t = 0$

$$y(t) = \sum_{i=0}^{\infty} y^{(i)}(0) \frac{t^i}{i!}$$

and its truncated Taylor expansion $y_N(t) = \sum_{i=0}^{N} y^{(i)}(0) \frac{t^i}{i!}$ at order $N$. 
The usual rules of symbolic calculus in Schwartz’s distribution theory [Schwartz 66] yield

\[ y_N^{(N+1)}(t) = y(0)\delta^{(N)} + \ldots + y^{(N)}(0)\delta, \]

where \( \delta \) is the Dirac measure at zero. Multiply both sides by \((-t)^i\)

\[ (-t)^i y_N^{(N+1)}(t) = (-t)^i \left( y(0)\delta^{(N)} + \ldots + y^{(N)}(0)\delta \right) \]

and apply the rules \( t\delta = 0, \ t\delta^{(i)} = -i\delta^{(i-1)}, \ i \geq 1 \). We obtain a triangular system of linear equations from which the derivatives \( y^{(i)}(0) \) can be obtained (\( 1 \leq i \leq N \))

\[ (-t)^i y_N^{(N+1)}(t) = \frac{N!}{(N-i)!} \delta^{(N-i)}y(0) + \ldots + \delta y^{(N-i)}(0) \quad (37) \]

It means that the coefficients \( y(0), \ldots, y^{(N)}(0) \) are linearly identifiable [Fliess et al. 04, Fliess et al. 08].
The time derivatives of $y_N(t)$, the Dirac measures and its derivatives are removed by integrating with respect to time both sides of Eq. (37) at least $\nu$ times ($\nu > N$):

$$
\int_0^t \int_0^{t\nu-1} \cdots \int_0^{t_1} (-\tau)^i y_N^{(N+1)} dt_\nu \cdots dt_1 d\tau = \frac{N!}{(N-i)!} \frac{t^{\nu-N-i-1}}{\nu-N-i-1)!} y(0) + \cdots + \frac{t^{\nu-1}}{(\nu-1)!} y^{(N-i)}(0)
$$

The iterated integrals may be replaced by

$$
\int_0^t \int_0^{t\nu-1} \cdots \int_0^{t_1} \tau^\alpha x(\tau) dt_\nu \cdots dt_1 d\tau = \int_0^t \frac{(t-\tau)^{\nu-1}}{(\nu-1)!} \tau^\alpha x(\tau) d\tau.
$$
It is clear that the numerical estimation rely on
\[ \lim_{N \to +\infty} \left[ y_N^{(i)}(0) \right]_{\text{estim}}(t) = y^{(i)}(0). \]

**Remark**

*These iterated integrals are low pass filters which attenuate the noises, which are viewed as highly fluctuating phenomena (see [Fliess 06] for more details).*

**Remark**

*The above formulae may easily be extended to sliding time windows in order to obtain real time estimates (see [Mboup et al. 07] for further details).*
Estimation

M. Fliess and H. Sira-Ramírez. 
*Identification of Continuous-time Models from Sampled Data*, chapter Closed-loop parametric identification for continuous-time linear systems via new algebraic techniques. 

M. Fliess.

Analyse non standard du bruit. 

Compression différentielle de transitoires bruités. 
Estimation


Estimation

C. Nöthen.
Beiträge zur rekonstruktion nicht direkt gemessener größen bei der silizium-einkristallzüchtung nach dem czochralski-verfahren.

L. Schwartz.
*Théorie des distributions. 2nd ed.*
Hermann, 1966.
Estimation

Algorithm

Off line:

1. determine the maximum number of output derivative necessary for estimation (take the highest derivative of the output in the collection of LODE describing the switching system),

2. test distinguishability using conditions given in (11) which will provide the “bad” input, let us note that the second relation of (11) can be used to check if the input is a “bad” one just by checking if it satisfies the differential relation.
Estimation
Algorithm

**On line:**

1. using Alien technics (see before) compute $y, \dot{y}, \ldots, y^{(k y_{\text{max}})}; u, \dot{u}, \ldots, u^{(k u_{\text{max}})}$,

2. check if $r_i(t)$ is zero for some time interval then the corresponding active subsystem is the “$i$-th”;

3. deduce the continuous state estimate using 1.
Breakthrough work of M. Fliess which takes an algebraic viewpoint shows a criterion for observability using a module-theoretic framework.

Theorem (M. Fliess et al.)

Observability of a system \((ODE: \text{linear or not})\) is equivalent to the possibility to express all the state variables of the system, (in particular all the state variables) as combinations of the components of the input, the output and of their time derivatives up to a finite order.
Let us assume that all submodels are observable: this means that for each submodel the state $x$ is a combination of $(y, u)$ and a finite number of their derivatives:

$$x(t) = \left( \begin{array}{c} \Gamma_0 \\ \Gamma_1 \\ \Gamma_2 \\ \vdots \\ \Gamma_{n_i-1} \end{array} \right) - M \left( \begin{array}{c} u \\ \dot{u} \\ u^{(2)} \\ \vdots \\ u^{(n_i-2)} \end{array} \right)$$

(39)
Estimation : State variable estimation

Two ways

\[
\begin{align*}
y &= C_i x \\
\dot{y} &= C_i \dot{x} = C_i (A_i x + B_i u) \\
\ddot{y} &= C_i A_i (A_i x + B_i u) + C_i B_i \dot{u} \\
\ldots &= \ldots \\
y^{(n_i)} &= C_i A_i^{(n_i)} x + C_i A_i^{(n_i-1)} B_i u + \ldots + C_i B_i u^{(n_i-1)}
\end{align*}
\]
Estimation : State variable estimation

Two ways

1. use the i/o relation and some algebraic manipulation in order to estimate the derivatives,
2. on use polynomial time serie of $y(t)$ on a time window, plus some algebraic estimation in order to get these derivatives (this is already done : we use it to estimate $\sigma$)
Example

Let us consider the following switching system (8) where

\begin{align*}
    i &= 1: \ddot{y} + y = u; & & \begin{align*}
    \dot{x}_1 &= -x_1 + u \\
    y &= x_1 \\
    \dot{x}_1 &= x_2 \\
    \dot{x}_2 &= -x_1 - x_2 + u \\
    y &= x_1 + x_2
    \end{align*} \\
    i &= 2: \ddot{y} + \dot{y} + y = \dot{u} + u; & & \begin{align*}
    \dot{x}_1 &= -\frac{1}{2}x_1 + u \\
    y &= x_1 \\
    \dot{x}_1 &= x_2 \\
    \dot{x}_2 &= -2x_1 - x_2 + u \\
    y &= x_1 + x_2 \\
    i &= 3: 2\dot{y} + y = 2u \\
    & \begin{align*}
    \dot{x}_1 &= -\frac{1}{2}x_1 + u \\
    y &= x_1 \\
    \dot{x}_1 &= x_2 \\
    \dot{x}_2 &= -2x_1 - x_2 + u \\
    y &= x_1 + x_2
    \end{align*} \\
    i &= 4: \ddot{y} + \dot{y} + 2y = \dot{u} + u;
\end{align*}

\text{Alien Groupe SdH, 2 Octobre, 2008, Paris, France}
Example

The following table gives the singular input/output for which distinguishability is lost.

<table>
<thead>
<tr>
<th>$i \backslash j$</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$X$</td>
<td>$\begin{cases} u = u_0 \ y = u_0 \end{cases}$</td>
</tr>
<tr>
<td>2</td>
<td>$\begin{cases} u = u_0 \ y = u_0 \end{cases}$</td>
<td>$X$</td>
</tr>
<tr>
<td>3</td>
<td>$\begin{cases} u = 0 \ y = 0 \end{cases}$</td>
<td>$\begin{cases} u = u_0 \exp(t) \ y = u_0 \exp(t) \end{cases}$</td>
</tr>
<tr>
<td>4</td>
<td>$\begin{cases} u = u_0 \exp(-t) \ y = u_0 \exp(-t) \end{cases}$</td>
<td>$\begin{cases} u = u_0 \exp(-t) \ y = 0 \end{cases}$</td>
</tr>
</tbody>
</table>
## Example

<table>
<thead>
<tr>
<th>$i \backslash j$</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>{ $u = u_0$  \ $y = u_0$ }</td>
<td>{ $u = u_0 \exp(-t)$  \ $y = u_0 \exp(-t)$ }</td>
</tr>
<tr>
<td>2</td>
<td>{ $u = u_0 \exp(t)$  \ $y = u_0 \exp(t)$ }</td>
<td>{ $u = u_0 \exp(-t)$  \ $y = 0$ }</td>
</tr>
<tr>
<td>3</td>
<td>X</td>
<td>{ $u = u_0 \exp(3t)$  \ $y = u_0 \exp(3t)$ }</td>
</tr>
<tr>
<td>4</td>
<td>{ $u = u_0 \exp(3t)$  \ $y = u_0 \exp(3t)$ }</td>
<td>X</td>
</tr>
</tbody>
</table>
Example

For the first order systems, in that follows, $x_2$ is enforced to zero. Moreover, the output continuity is ensured between two systems whereas initial condition of derivative output is randomly chosen in $[-0.5, +0.5]$. Residuals associated to previous systems are

\[
\begin{align*}
  i = 1: r_i &= [\dot{y}]_e + [y]_e - u \\
  i = 2: r_i &= [\ddot{y}]_e + [\dot{y}]_e + [y]_e - [\dot{u}]_e - u \\
  i = 3: r_i &= 2[\dot{y}]_e + [y]_e - 2u \\
  i = 4: r_i &= [\ddot{y}]_e + [\dot{y}]_e + 2[y]_e - [\dot{u}]_e - u
\end{align*}
\]

where $[\bullet]_e$ is the estimation of $\bullet$ and to $[y]_e$ corresponds the $y$ denoised signal.

Without noise, output derivatives are estimated according to the well known Euler’s method.
Example
Free noise results: constant input

Systems 1 and 2 are indistinguishable for $y_0 = u_0 = 1$, i.e. $r_1 = r_2 = 0$. 

![Diagram showing the switching signal]
Example

Free noise results: constant input

**Figure:** Output (−); input (−−)
Example

Free noise results: constant input

**FIG.:** Residuals: $|r_1|$ (−); $|r_2|$ (−−); $|r_3|$ (··); $|r_4|$ (−·)
Example

In the next figure, system distinguishability is easy and ensures a very good state estimation.

Fig.: Switching signal $\sigma$
Example

Free noise results: sinusoidal input

**Figure:** Output (−); input (−−)
Example

Free noise results: sinusoidal input

![Diagram showing residuals](image-url)

**Fig.:** Residuals: $|r_1| (-)$; $|r_2| (- -)$; $|r_3| (..)$; $|r_4| (- .)$
Example

Free noise results: sinusoidal input

**Fig.** State: $x_1 (-); x_2 (- -); [x_1]_e (\cdot \cdot); [x_2]_e (- \cdot)
In noisy case (additive output noise $N(0, 0.01)$), Euler’s method is not available.

Apply recent results on derivative estimation (see [Mboup et al. 07]) in order to evaluate residuals. They are approximatively null when the associated system is active and becomes non zero in other case. However, to take the decision, that is to say to know what is the active system, is not easy (see figure ??-(c)). Here, the mean of each residual is calculated along a sliding window. Thus at the smallest mean of residual is associated the active system. According to this logic, states are estimated.
Example

Noised results: sinusoidal input

**FIG.**: Switching signal $\sigma$
Example
Noised results: sinusoidal input

**FIG.**: Output (–); input (---)
Example

Noised results: sinusoidal input

Fig.: Residuals: $|r_1|$ (-); $|r_2|$ (- -); $|r_3|$ (. .); $|r_4|$ (- .)
Example

Noised results: sinusoidal input

\[ \text{Fig.: State: } x_1 (-); x_2 (- -); [x_1]_e (\ldots); [x_2]_e (- .) \]
Example

In the previous figures, rather than to estimate output derivative in real time, a small constant and known delay is allowed for estimations (see [Mboup et al. 07] for more details). In this case, in exactly the same simulation context than previously, decision according to residuals is easier.
Example

Noised results: sinusoidal input and delayed estimations

**Fig.** Residuals: $|r_1| (-)$; $|r_2| (---)$; $|r_3| (.)$; $|r_4| (-.)$
Example

Noised results: sinusoidal input and delayed estimations

**Fig.:** State: $x_1 (-)$; $x_2 (- -)$; $[x_1]_e (\cdot \cdot)$; $[x_2]_e (- \cdot \cdot)$
Example
Noised results : sinusoidal input and filtered estimations

Figure: Residuals : $|r_1|$ (–); $|r_2|$ (- -); $|r_3|$ ( . .); $|r_4|$ ( - .)
Example

Noised results: sinusoidal input and filtered estimations

**Fig.:** State: $x_1$ (-); $x_2$ (- -); $[x_1]_e$ ( . . ); $[x_2]_e$ (- . )