Modeling hybrid linear systems with Bond-Graph using an implicit formulation

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Abstract - This paper deals with the modeling of linear hybrid physical systems. The bond graph approach is used to establish the properties of a knowledge model. Ideal switching components which modify the circuit topology at switching time are represented by flow or effort sources according to their state. The paper shows the interest of the implicit representation: - to derive a unique implicit state equation with jumping parameters, - to derive the implicit state equation with index of nilpotency one corresponding to each configuration, - to analyse the properties of those models and - to compute the transitions.

Keywords : Bond-graph, Hybrid systems, Implicit systems, Descriptor systems

I. Introduction

More and more interest is given to the study of hybrid systems which are composed of interacting continuous and discrete parts. Different models are proposed [2]. Generally, the discrete part is modeled by automata, and the continuous part is modeled by different sets of state equations according to the discrete state. Jump functions can be added to represent the discontinuities. But no indication is given about the relations between the different models and the different jump functions.

In this paper the bond-graph approach is used to derive models for physical systems with switching devices. Switching devices which change the structure of systems are very common in the field of electricity (e.g., diodes, relays ..) and also in other fields (e.g., clutches, valves ...). They are the interface between the discrete part (control) and the physical continuous part.

Various researchers have studied this problem, and there are two main approaches : ideal switches with variable circuit topology [3,4,5] and non-ideal switches with constant circuit topology [6]. This second approach is a way to solve the problems for the simulation but is not applicable to the study of the model of systems with ideal switches. The first approach which is used here, permits both a simplified analysis with ideal elements when considering the global performance of the system and a more precise analysis if a more realistic model of the switching device is known.

The paper is organized as follows: in section II the notion of implicit junction structure matrices for switching bond-graphs is introduced. In section III the state equation is derived from this representation. In section IV the properties of the model are studied and the discontinuities are computed. In section V, transition matrices are proposed to compute the new implicit junction structure matrixes after a commutation. Finally a simple example illustrates the different points.

II. An implicit formulation of the junction structure relation

With the bond-graph technique ideal switches can be modeled by zero effort (flow) sources when they are in one state and by zero flow (effort) sources when they are in a second state. If the switch is not ideal, it can be modeled by an ideal switch associated with other elements (R, I, C) on a 1 or 0 junction. When the switches commutate, the causality of the corresponding sources changes and it is necessary to extend this change of causality to other bonds. Some energy storage elements can lose or recover the integral causality and the state equation is changed.

A general bond-graph can always be represented by the diagram of figure 1 where a field with the switching components is distinguished. The convention for the direction of the power is shown on figure 1 i.e. from the sources towards the junction structure or from the junction structure towards the I, C and R elements.

![Figure 1](http://www.ece.arizona.edu/~cellier/bg_digest.html)

The causality assignment can be done for any initial acceptable configuration of the switches. The bond-graph is then equivalent to the block diagram represented on figure 2.
where the following key variables are used: 
- the state vector $X_t$ is composed of the energy variables in integral causality ($p$ on I elements and $q$ on C elements), and the complementary state vector $Z_d$ is composed of power variables ($e$ on C elements and $f$ on I elements); 
- the vector called semi-state vector $X_s$ is composed of the energy variables in derivative causality, ($p$ on I elements and $q$ on C elements), and the complementary state vector $Z_d$ composed of power variables ($e$ on C elements and $f$ on I elements); 
- the vectors $D_i$ and $D_o$ represent the variables going out of and into the $R$ field; 
- the vector $U$ is composed of the sources; 
- $T_e$ is composed of the variables imposed by the switches in this configuration: the flow for the switches which are opened and the effort for the switches which are closed; 
- $T_o$ is composed of the variables in the switches: the effort for the switches which are opened and the flow for the switches which are closed.

If we suppose that there is no causal loop [8] with a unity gain, we can relate each output of the junction structure ($X_s$, $Z_d$, $D_i$ and $T_o$) as function of all the inputs ($X_s$, $Z_d$, $D_i$ and $T_o$) with the following implicit equation that we call the standard implicit form:

$$ M\dot{X} = SW $$

with

$$ W = \begin{pmatrix} Z_i & Z_d & D_i & D_o & T_e & T_o & U \end{pmatrix}^T $$

$$ M = \begin{pmatrix} 1 & -S_{12} \\ 0 & 0 & 0 & 0 & S_{11} & 0 & 0 \\ 0 & 0 & -S_{23} & -1 & 0 & 0 & S_{24} & 0 & S_{23} & 0 & S_{25} \\ 0 & 0 & -S_{34} & 0 & S_{33} & -1 & 0 & S_{34} & 0 & S_{35} \\ 0 & 0 & -S_{45} & S_{46} & 0 & S_{44} & 0 & S_{45} & -1 & S_{46} & 0 \end{pmatrix} $$

(2)

where 1 and 0 denotes structural identity and zero matrices, and the other elements are composed of $0, \pm 1$, and the coefficient of the gyrators and transformers. The first row of this relation defines $\dot{X}_i$, the second one $Z_d$, the third one $D_i$, and the last one $T_o$.

$S_{11}$, $S_{33}$ and $S_{44}$ are skew symmetric. Those properties are due to energetic properties (there is no power stored or generated in the junction structure). There is no relation between $Z_d$ and $D_i$ (or $D_o$ and $\dot{X}_i$). If it were the case, we could inverse a causal path between an element in derivative causality and a resistor port and give to this element the integral causality. For the same reason, there is no relation between $X_s$ and $Z_d$.

At this point it is very important to note that:

- As the formal relations are not changed, the implicit equation that was found for one configuration of the switches is however true whatever the state of the switches is as long as no value is affected to $T_e$ or $T_o$.

As we will see in the next section this expression permits to compute a unique implicit state equation. If the system is not in the initial configuration, the elements of $T_e$ corresponding to the switches which are not in their initial state will no longer be null but the corresponding elements of $T_o$ will be null.

- For the different configurations of the switches, if the system is solvable, such a standard implicit form can be found. Some elements of the vectors $W$ and $X$ can permute, and the elements of the matrices $M$ and $S$ change, but the structural zero ($0$) or identity ($I$) matrices will always be the same. As it will be seen in section IV, this form is really convenient to compute the state equation in the initial configuration because in the ideal case $T_e = 0$.

### III. Unique State equation formulation

In the linear case, the third line permits to solve analytically the algebraic loop between the resistive elements ($S_{33} \neq 0$). If the constitutive law for the $R$ field is $D_i = LD$, and admitting that $(I - S_{33} I)$ is invertible, then

$$ D_e = (I - S_{33} I)^{-1} (-S_{33} Z_i + S_{34} T_e + S_{35} U) $$

The constitutive relation for the inertial and capacitive fields

$$ \begin{pmatrix} Z_i \\ Z_d \end{pmatrix} = \begin{pmatrix} F & F \\ F^T & F \end{pmatrix} \begin{pmatrix} X_i \\ X_s \end{pmatrix} $$

is used to eliminate $Z_i$ and $Z_d$, and denoting $H = L(I - S_{33} I)^{-1}$ and $K = (S_{11} - S_{13} H S_{31}^T)$, we get the implicit equation:

$$ \begin{pmatrix} 1 & -S_{12} \\ 0 & 0 & -S_{34} \end{pmatrix} \begin{pmatrix} \dot{X}_i \\ \dot{X}_s \end{pmatrix} = \begin{pmatrix} S_{11} - S_{13} H S_{31}^T & 0 & S_{13} + S_i H S_{31} & 0 \\ 0 & 0 & -S_{34} \end{pmatrix} \begin{pmatrix} Z_i \\ Z_d \end{pmatrix} + \begin{pmatrix} S_{31} + S_{13} H S_{31} \\ S_{31} + S_{13} H S_{31} \end{pmatrix} U $$

$$ + \begin{pmatrix} S_{13} + S_i H S_{31} \\ S_{13} + S_i H S_{31} \end{pmatrix} \begin{pmatrix} Z_i \\ Z_d \end{pmatrix} $$

Which is not a state equation due to the presence of $T_e$ and $T_o$ whose entries are inputs or outputs according to the configuration of the switches. To obtain a state
equation a fourth line has to be added to take account of

\[
\begin{pmatrix}
1 & -S_{12} & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & -S_{12}^T & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
\dot{X}_1 \\
\dot{X}_2 \\
\dot{X}_3 \\
\dot{X}_4 \\
\end{pmatrix}
= 
\begin{pmatrix}
K.F_i & KF & S_{14} + S_{13}.H.S_{34} \\
- S_{12}^T.F_i - F^T & - S_{12}^T.F - F_d & S_{24} \\
(S_{14} - S_{13}.H.S_{13}^T).F_i & 0 & S_{44} + S_{14}^T.H.S_{34} - I \\
0 & 0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
X_1 \\
X_2 \\
X_3 \\
X_4 \\
\end{pmatrix}
+ 
\begin{pmatrix}
S_{15} + S_{13}.H.S_{35} \\
S_{45} + S_{34}.H.S_{35} \\
\end{pmatrix}
\begin{pmatrix}
U \\
\end{pmatrix}
\tag{4}
\]

In the initial configuration \( TT = 0 \) and the fourth line leads to \( T = 0 \), in another mode, \( TT \) is a diagonal matrix whose elements are all null except those corresponding to the switches which are in a different mode than in the initial configuration and the fourth line leads to \( (1 - TT)T = 0 \) and \( TT.T_0 = 0 \).

The advantage of this form is that it is very synthetic. The commutating system is modeled by a unique state equation with a jump on a small part of its parameters (\( TT \)). However, the index of nilpotency is greater than one and this form is not convenient for simulation.

### IV. Properties of the state model

In this part, it is supposed that the system is in one configuration, and the properties of the implicit state equation are studied as well as the transition with the previous configuration.

If the implicit form (1) has been established for that configuration, \( T = 0 \) and the state equation (4) can be simplified:

\[
\begin{pmatrix}
1 & -S_{12} & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & -S_{12}^T & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
\dot{X}_1 \\
\dot{X}_2 \\
\dot{X}_3 \\
\dot{X}_4 \\
\end{pmatrix}
= 
\begin{pmatrix}
K.F_i & KF & S_{14} + S_{13}.H.S_{34} \\
- S_{12}^T.F_i - F^T & - S_{12}^T.F - F_d & S_{24} \\
(S_{14} - S_{13}.H.S_{13}^T).F_i & 0 & S_{44} + S_{14}^T.H.S_{34} - I \\
0 & 0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
X_1 \\
X_2 \\
X_3 \\
X_4 \\
\end{pmatrix}
+ 
\begin{pmatrix}
S_{15} + S_{13}.H.S_{35} \\
S_{45} + S_{34}.H.S_{35} \\
\end{pmatrix}
\begin{pmatrix}
U \\
\end{pmatrix}
\tag{5}
\]

which is a standard implicit equation

\[
EX = AX + BU
\tag{6}
\]

This model has been studied by several authors [7,9]. It is an association of dynamical and algebraic modes and it can exhibit impulsive and discontinuous phenomena at the origin. One of the main results is the following:

- if \( \text{dim}(X) = n \), and \( \text{rank}(E) = q < n \), then there are \( q \) dynamical modes and \( n - q \) algebraic modes. Moreover, if

\[
\begin{pmatrix}
E \\
0 \end{pmatrix}
\]

its rank is \( n + q - h \) then among the dynamical modes, there are \( q - h \) finite modes and \( h \) infinite modes.

It is usual when analysing the properties of (6) to premultiply it by a non singular matrix \( P \), and to operate a variable change \( Q \) to obtain an equivalent equation:

\[
(PEQ)(Q^{-1}\dot{X}) = PAQ(Q^{-1}X) + (PB)U
\]

To go further in the analysis of the implicit equation, let's premultiply (5) by the non singular matrix

\[
P = \begin{pmatrix}
1 & -K.(F + F_i.S_{12}) & 0 \\
0 & I & 0 \\
0 & 0 & -R \\
\end{pmatrix}
\]

the constraint on the switches:

\[
R = (S_{12}^T.F_i.S_{12} + F^T.S_{12} + S_{12}.F + F_d)^{-1}
\]

leads to the positivity of \( F_i \) and \( F_d \), and let do the variable change

\[
Q = \begin{pmatrix}
1 & S_{12} \\
I & 0 \\
0 & -R.(S_{12}^T.F_i + F^T) \\
\end{pmatrix}
\]

We get the following implicit state equation:

\[
\begin{pmatrix}
1 & 0 \\
0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
\dot{X}_1 \\
\dot{X}_2 \\
\end{pmatrix}
= 
\begin{pmatrix}
K.F_i - K.(F + F_i.S_{12}).B.(S_{12}^T.F_i + F^T) & 0 \\
S_{15} + S_{13}.H.S_{35} + K.(F + F_i.S_{12}).R.S_{35} \\
-R.S_{35} \\
\end{pmatrix}
\begin{pmatrix}
X_1 \\
X_2 \\
\end{pmatrix}
+ 
\begin{pmatrix}
S_{15} + S_{13}.H.S_{35} + K.(F + F_i.S_{12}).R.S_{35} \\
-R.S_{35} \\
\end{pmatrix}
\begin{pmatrix}
U \\
\end{pmatrix}
\tag{7}
\]

which shows that this implicit state equation with an index of nilpotency equal to one is equivalent to an ordinary state equation:

\[
\dot{X}_i = (K.F_i - K.(F + F_i.S_{12}).B.(S_{12}^T.F_i + F^T))X_i
+ (S_{15} + S_{13}.H.S_{35} + K.(F + F_i.S_{12}).R.S_{35})U
\]

where the state \( X_i \) is continuous at the origin, associated to an algebraic equation:

\[
X_i = R.S_{35}.U
\]

This form allows to verify that in the case of hybrid physical systems, \( h = 0 \) which implies that state variables are not impulsive on commutation, which is coherent with the physicals principles.

**A. Discontinuities on the variables at the commutation**

At the initial time which is the commutation time, \( X_i^+ = X_i^- \) and \( X_i^+ = R.S_{35}.U \).

Using the previous variable change \( Q \), it is easy to compute the discontinuity on the original state variables:

\[
X_i^+ = X_i^- + S_{12}.(X_d^- - X_d^+)
\tag{8}
\]

and

\[
X_d^+ = R.((S_{12}^T.F_i + F^T).S_{35}X_i^- + S_{35}.U)
\tag{9}
\]

**B. Amplitude of the pulses at the commutation**

Another approach to compute what happens on a commutation is to use the implicit equation (1) or (3) of the system after the commutation. In this equation, \( T_0 \) can be impulsive on commutation:

- a switch which doesn’t commutate can have an impulsive flow if it was closed (and its effort stays null) or an impulsive effort if it was opened (and its flow stays null);
- a switch which commutates can have an impulsive flow if it becomes closed (and its effort becomes null) or an
impulsive effort if it becomes opened (and its flow
becomes null),
so that \( T_0 = T_0 \delta(t - t_1) \), and the integration of the first and
two third lines gives :

\[
\begin{bmatrix}
1 & -S_{i_6} \\
0 & -S_{i_6}'
\end{bmatrix}
\begin{bmatrix}
X^+ - X^- \\
X^+_t - X^-_t
\end{bmatrix} = \begin{bmatrix} 0 \\ -T \end{bmatrix}
\]

and the second line is the constraint

\[-S_{i_6}' Z^+_j - Z^+_j + S_{i_6} U = 0 \]

which gives the same result for the discontinuity, and the
amplitude of the pulses in the switches :

\[ T = S_{i_6}' (X^+_t - X^-_t) \]

V. The transition matrix

The preceding section has shown the interest of the
standard implicit junction structure form which allows to
find easily the state equation in the corresponding
configuration and to compute the discontinuities. This
section examines how to recover this form after a
commutation. The case with the commutation of only one
switch is first considered. When a switch commutates, there
are 3 possibilities : - one element loses its integral causality,
- one element recovers its integral causality, - there is no
change of causality in the energy storage elements. The first
case is described in details in the following; the two other
cases are describe in the appendix.

A. Worked case when one element loses its integral
causality

Let's consider the case of a system with one element losing
its integral causality on commutation. Reordering the vectors
and matrixes, we can write :

\[
\begin{bmatrix}
1 & 0 & -S_{i_6} \\
0 & 1 & -S_{i_6} \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & -S_{i_6}' \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\ddot{X}_t \\
\dot{X}_t \\
X_t \\
X^+_t \\
X^-_t
\end{bmatrix} =

\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
Z_x \\
Z_e \\
S_{i_6} \\
S_{i_6}' \\
T_x \\
T_e
\end{bmatrix}
\]

\[
\begin{bmatrix}
S_{i_6} & S_{i_6}' & 0 & S_{i_6} & 0 & 0 & c_{i_6} & S_{i_6} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-S_{i_6} & -S_{i_6}' & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -I & S_{i_6} & 0 & 0 & 0 & 0 \\
S_{i_6}' & I_{i_6} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -c_{i_6} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
Z_x \\
Z_e \\
S_{i_6} \\
S_{i_6}' \\
T_x \\
T_e
\end{bmatrix}
\]

where \( T_1 \) and \( T_2 \) are the variables associated to the
switch which commutates, and \( X_{i_6d} \) is the power variable
associated to the element which will lose its integral causality.

As one element loses its integral causality, there is no
relation between \( D_x \) and \( T_1 \), and there is a causal path

between \( \dot{X}_{i_6d} \) and \( t_1 \) so that \( e_{i_6} \neq 0 \). There is no more
relation between \( Z_{i_6} \) and \( t_1 \) (or \( t_2 \) and \( \dot{X}_{i_6d} \)), in the other case
one element would recovered the integral causality on
commutation.

It is this seventh column of \( S \) which indicates the elements in causal relation with the switch. If the third line
were not zero, one element would have recover its integral
causality; if the third line was zero and the fourth line was
not zero, a resistive element would have exchanged its
causality with the switch. It is because those two elements
are null, and \( e_{i_6} \neq 0 \), that the corresponding element gets
the derivative causality.

As we have supposed that it is the last element of \( X_j \)
which loses its integral causality, and if we decide to put this
element on top of \( X_{i_6} \), then after exchanging \( t_1 \) and \( t_2 \) we obtain :

\[
\begin{bmatrix}
1 & 0 & -S_{i_6} \\
0 & 1 & -S_{i_6} \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & -S_{i_6}' \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\dot{X}_t \\
\ddot{X}_t \\
X_t \\
X^+_t \\
X^-_t
\end{bmatrix} =

\begin{bmatrix}
Z_x \\
Z_e \\
S_{i_6} \\
S_{i_6}' \\
T_x \\
T_e
\end{bmatrix}
\]

\[
\begin{bmatrix}
S_{i_1} & S_{i_5} & 0 & S_{i_4} & 0 & 0 & c_{i_4} & S_{i_5} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-S_{i_2} & -S_{i_2}' & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -I & S_{i_4} & 0 & 0 & 0 & 0 \\
S_{i_4}' & I_{i_4} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -c_{i_4} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
Z_x \\
Z_e \\
S_{i_6} \\
S_{i_6}' \\
T_x \\
T_e
\end{bmatrix}
\]

which has lost the structural 1’s and 0’s of the standard
form. If we multiply each side of the preceding relation by the
nonsingular matrix :

\[
V =
\begin{bmatrix}
1 & -s_{i_4} & 0 & 0 & 0 & -c_{i_4} \\
0 & 0 & 0 & 0 & 0 & -e_{i_4} \\
0 & s_{i_3} & I & 0 & 0 & 0 \\
0 & s_{i_3}' & 0 & I & 0 & 0 \\
0 & -I_{i_3} & 0 & 0 & I & -s_{i_4} \\
0 & -e_{i_3} & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
I & -s_{i_4} & 0 & 0 & 0 & -s_{i_4}' \\
0 & 0 & 0 & 0 & 0 & -1/e_{i_4} \\
0 & s_{i_3} & I & 0 & 0 & 0 \\
0 & s_{i_3}' & 0 & I & 0 & 0 \\
0 & -I_{i_3} & 0 & 0 & I & -s_{i_4} \\
0 & -e_{i_3} & 0 & 0 & 0 & 0
\end{bmatrix}^{-1}
\]

\[
=\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0
\end{bmatrix}
\]

where \( V^{-1} \) is directly composed with columns in (10) :

column 1 from the matrix in the left part and columns 2,3,5,
8 and 9 from the matrix in the right part. We obtain the new
expression which has the standard implicit form :

\[
\begin{bmatrix}
I & -s_{i_4} & 0 & 0 & 0 & -s_{i_4}' \\
0 & 0 & 0 & 0 & 0 & -1/e_{i_4} \\
0 & s_{i_3} & I & 0 & 0 & 0 \\
0 & s_{i_3}' & 0 & I & 0 & 0 \\
0 & -I_{i_3} & 0 & 0 & I & -s_{i_4} \\
0 & -e_{i_3} & 0 & 0 & 0 & 0
\end{bmatrix}^{-1}
\]

\[
=\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0
\end{bmatrix}
\]
The same technique can be used in a straightforward way to compute the transition matrices and the junction structure matrices after commutation when there is no change of integral causality on commutation, or when an element recovers its integral causality.

As the structural relation (which is not unique) depends only on the switches configuration, but not on the way to reach this configuration, the case of $N$ switches commutating at the same time, can be solved as $N$ consecutive commutations of one switch.

It is the examination of column seventh which indicates the case. The possibility of an element recovering its integral causality is first checked, then the possibility of no change of integral causality, and finally the possibility of one element losing its integral causality. If there is no solution, it reveals a causality conflict.

VI. Example

Let us consider the electromechanical system (Figure 3) composed with two DC motors coupled by an ideal clutch. When the motors are coupled, their angular velocities are identical and when they are decoupled, they can be considered as two independent systems. The bond-graph model is shown on Figure 4. To simplify the presentation, the inductances and electrical resistances are set equal to zero, and the motors are controlled by an input current. The clutch is represented by the X element. The flows $f_{11}$ and $f_{21}$ represent the angular speeds.

![Figure 3. An electromechanical system](image)

Figure 4 the associated bond-graph.

If the clutch is perfect, when it is let out, the motors are not coupled (mode 1), it can be modeled by a zero effort source. No effort is transmitted on the shaft. When it is let in, the motors are coupled (mode 2), the speed of the two motors are identical (with the notations of the graph, the sign is different). It can be modeled as a zero flow source. Let’s consider mode 2 as the reference mode. During the assignment procedure, when we apply the necessary causality to the bond 4 and extending the causal implication through the 0-junction and the 1-junctions, one of the I elements must have the derivative causality.

![Figure 5 Causal bond-graph when the motors are coupled (mode 2)](image)

For this causal bond-graph, the implicit equation is :

\[
\begin{bmatrix}
1 & 1 \\
0 & 0 \\
0 & 0 \\
0 & -1 \\
\end{bmatrix}
\begin{bmatrix}
\hat{p}_{11} \\
\hat{p}_{21} \\
\hat{p}_{12} \\
\hat{p}_{22} \\
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
-1 \\
0 \\
\end{bmatrix}
\]

A. Transition matrix

After a commutation, consisting in exchanging columns 7 and 8, the standard implicit form can be recovered by multiplying the previous equation by $V$ derived from (11):
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When the motors are coupled for each configuration. The first one, when the motors are coupled and \( t_2 = 0 \) so that \( t_1 = 0 \), is directly derived by using the 2 fist lines :

\[
\begin{bmatrix}
1 & 1 \\
0 & 0 \\
0 & -1 \\
0 & 0 \\
\end{bmatrix} \begin{bmatrix}
\dot{p}_{11} \\
\dot{p}_{21} \\
\end{bmatrix} =
\begin{bmatrix}
-f_{11} \\
f_{21} \\
e_{12} \\
e_{22} \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 0 & -1 & 0 & 0 & -1 & 0 & K_1 & 0 \\
0 & 0 & 0 & -1 & 0 & 1 & 0 & K_2 \\
1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
T_1 \\
T_2 \\
I_1 \\
I_2 \\
\end{bmatrix}
\]

which is the standard implicit junction structure relation in mode 1.

\textbf{B. State equations}

One implicit general state equation can be deduced :

\[
\begin{bmatrix}
1 & 1 \\
0 & 0 \\
-1 & 0 \\
0 & 0 \\
\end{bmatrix} \begin{bmatrix}
\dot{p}_{11} \\
\dot{p}_{21} \\
\end{bmatrix} =
\begin{bmatrix}
-f_{11} \\
f_{21} \\
e_{12} \\
e_{22} \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
-b_1 / J_1 & -b_2 / J_1 & -b_2 & 0 & K_1 & K_2 & 0 \\
1 / J_1 & -1 / J_2 & 1 & 0 & 0 & 0 & 0 \\
-b_2 / J_1 & 0 & b_2 & -1 & 0 & -K_2 & 0 \\
0 & 0 & 1 & u & u & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
T_1 \\
T_2 \\
I_1 \\
I_2 \\
\end{bmatrix}
\begin{bmatrix}
p_{11} \\
p_{21} \\
\end{bmatrix}
\]

This state equation is valid for both configurations. However, more simple distinct state equations can be found for each configuration. The first one, when the motors are coupled and \( t_2 = 0 \) so that \( t_1 = 0 \), is directly derived by using the 2 fist lines :

\[
\begin{bmatrix}
1 & 1 \\
0 & 0 \\
\end{bmatrix} \begin{bmatrix}
\dot{p}_{11} \\
\dot{p}_{21} \\
\end{bmatrix} =
\begin{bmatrix}
-f_{11} \\
f_{21} \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
-b_1 / J_1 & -b_2 / J_1 & -b_2 & 0 & K_1 & K_2 & 0 \\
1 / J_1 & -1 / J_2 & 1 & 0 & 0 & 0 & 0 \\
-b_2 / J_1 & 0 & b_2 & -1 & 0 & -K_2 & 0 \\
0 & 0 & 1 & u & u & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
T_1 \\
T_2 \\
I_1 \\
I_2 \\
\end{bmatrix}
\begin{bmatrix}
p_{11} \\
p_{21} \\
\end{bmatrix}
\]

When the motors are coupled \( u = 1 \) so that \( t_1 = 0 \), to eliminate \( t_2 \), we need to combine the first three lines, and we get :

\[
\begin{bmatrix}
1 & 0 \\
0 & 1 \\
\end{bmatrix} \begin{bmatrix}
\dot{p}_{11} \\
\dot{p}_{21} \\
\end{bmatrix} =
\begin{bmatrix}
-f_{11} \\
f_{21} \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
-b_1 / J_1 & 0 & K_1 & 0 & 0 \\
0 & -b_2 / J_2 & 0 & K_2 & 0 \\
\end{bmatrix}
\begin{bmatrix}
T_1 \\
T_2 \\
\end{bmatrix}
\begin{bmatrix}
p_{11} \\
p_{21} \\
\end{bmatrix}
\]

which could have been derived more easily from the standard implicit form in mode 1

\textbf{C. Computing discontinuities on commutation :}

Using the general formulas, we can compute for the transition from mode 1 to mode 2 :

\[
\begin{bmatrix}
1 & 1 \\
0 & -1 \\
\end{bmatrix} \begin{bmatrix}
p_{11}^- - p_{11}^+ \\
p_{21}^- - p_{21}^+ \\
\end{bmatrix} =
\begin{bmatrix}
0 \\
-1 \\
\end{bmatrix}
\]

and

\[
p_{11}^- - p_{11}^+ = 0, \quad p_{21}^- - p_{21}^+ = 0
\]

which finally gives :

- the amplitude of the effort pulse in the clutch
  \[ T = \frac{J_2 p_{11}^- - J_1 p_{21}^-}{(J_1 + J_2)} \]
- the variable after the commutation
  \[ p_{11}^+ = p_{11}^- - T \]
  \[ p_{21}^+ = p_{21}^- + T \]

which satisfies the principle of the conservation of momenta.

\textbf{VII. Conclusion}

This paper has shown some properties concerning the models of the continuous part of hybrid linear physical systems using the notion of implicit junction structure relation :

- A unique implicit state equation with jumping parameters has been proposed.
- When using a different equation for each configuration, it has been shown that the system can be modeled by non impulsive implicit linear state equations.
- An explicit solution has been proposed to compute variables on the commutation as well as the weight of the pulses which allows to determine the energy instantaneously lost at the commutation in the switches.
- A solution has been proposed to compute the implicit junction structure and then the implicit state equation for each mode.

Those two last points offer an alternative for the simulation of hybrid systems. Rather than using non linear elements to model switches, which can lead to numerical difficulties, it is more simple to use ideal components, with some simple extra formal computing to deduce the different models, and change the model on commutation. This method has been applied with success to the simulation of a large system consisting of an asynchronous motor associated to a starter [1].

\textbf{References}

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Appendix

A. Worked case when one element recovers its integral causality

Reordering the vectors and matrixes, we can write

\[
\begin{bmatrix}
    0 & 0 \\
    0 & 0 \\
    0 & 0 \\
    0 & 0 \\
    -I_{24} & 0 \\
    -e_{24} & -s_{24}
\end{bmatrix}
\begin{bmatrix}
    \dot{X}_d \\
    \dot{X}_d
\end{bmatrix}
= 
\begin{bmatrix}
    \mathbf{0} & -s_{12} & -s_{12} \\
    0 & 0 & 0 \\
    0 & 0 & 0 \\
    0 & 0 & 0 \\
    0 & -I_{24} & -s_{24} \\
    0 & -e_{24} & -s_{24}
\end{bmatrix}
\begin{bmatrix}
    \dot{X}_d \\
    \dot{X}_d
\end{bmatrix}
\begin{bmatrix}
    Z_e \\
    Z_e \\
    \frac{D_e}{D_e} \\
    \frac{D_e}{D_e} \\
    \frac{D_e}{D_e} \\
    \frac{D_e}{D_e}
\end{bmatrix}
= 
\begin{bmatrix}
    \mathbf{0} & S_{13} & S_{13} & 0 & 0 & S_{14} & c_{14} & 0 & 0 & S_{15}
\end{bmatrix}
\begin{bmatrix}
    S_{11} \\
    -s_{12} \\
    -I \\
    -S_{23} \\
    S_{14} \\
    c_{14}
\end{bmatrix}
\begin{bmatrix}
    Z_e \\
    Z_e \\
    \frac{T_e}{T_e} \\
    \frac{T_e}{T_e} \\
    \frac{T_e}{T_e} \\
    \frac{T_e}{T_e}
\end{bmatrix}
\begin{bmatrix}
    \frac{D_e}{D_e} \\
    \frac{D_e}{D_e} \\
    \frac{D_e}{D_e} \\
    \frac{D_e}{D_e} \\
    \frac{D_e}{D_e} \\
    \frac{D_e}{D_e}
\end{bmatrix}
\end{eqnarray}
\]

where \( t_1 \) and \( t_2 \) are the variables associated with the switch which commutates, and \( x_{id} \) is the power variable associated to the element which will recover its integral causality.  

As one element recovers its integral causality, there is a causal path between \( \dot{x}_{id} \) and \( t_1 \) so that \( e_{24} \neq 0 \). It is this seventh column which indicates the elements in causal relation with the switch.

As we have supposed that it is the first element of \( X_d \) which loses its integral causality, and if we decide to put this element beneath \( X_l \), then to recover the standard implicit form after exchanging \( t_1 \) and \( t_2 \) we have to multiply each side of the relation by the nonsingular matrix :

\[
V = \begin{bmatrix}
    1 & -s_{14} \\
    0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 \\
    0 & -I_{34} & 0 & 0 & -e_{34} & 0 \\
    0 & 0 & 0 & 0 & -e_{34} & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 
\end{bmatrix}^{-1}
\]

B. Worked case when no integral causality changes

Let’s consider the case of a system where the causality of one \( R \) element changes on commutation. Reordering the vectors and matrixes, we can write

\[
\begin{bmatrix}
    S_{11} & 0 & S_{13} & 0 & 0 & S_{14} & c_{14} & 0 & 0 & S_{15}
\end{bmatrix}
\begin{bmatrix}
    X_{id} \\
    X_{id}
\end{bmatrix}
= 
\begin{bmatrix}
    Z_e \\
    Z_e \\
    \frac{T_e}{T_e} \\
    \frac{T_e}{T_e} \\
    \frac{T_e}{T_e} \\
    \frac{T_e}{T_e}
\end{bmatrix}
\begin{bmatrix}
    D_e \\
    D_e \\
    \frac{D_e}{D_e} \\
    \frac{D_e}{D_e} \\
    \frac{D_e}{D_e} \\
    \frac{D_e}{D_e}
\end{bmatrix}
\end{eqnarray}
\]

where \( t_1 \) and \( t_2 \) are the variables associated to the switch which commutates, and \( d_{1} \) is in relation with \( t_2 \). If we suppose that it is the last element of \( D \) whose causality changes, after exchanging \( t_1 \) and \( t_2 \), and \( d_1 \) and \( d_2 \), then to recover the standard implicit form, we have to multiply each side of the relation by the nonsingular matrix :

\[
V = \begin{bmatrix}
    1 & 0 & 0 & -s_{14} \\
    0 & 1 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 \\
    0 & 0 & -I_{34} & 0 & -e_{34} \\
    0 & 0 & 0 & -e_{34} & 0 \\
    0 & 0 & 0 & 0 & 0 
\end{bmatrix}^{-1}
\]